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MULTIMODAL OPTIMIZATION OF ARCHES UNDER STABILITY CONSTRAINTS WITH TWO INDEPENDENT DESIGN FUNCTIONS

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Abstract—An elastic, plane and funicular circular arch loaded by uniformly distributed radial pressure is considered. Fundamental buckling modes corresponding to two lowest critical loads both for out-of-plane and in-plane buckling of the arch are studied. Both depth and width of a rectangular cross-section are treated as independent control functions. The optimization problem determines these cross-sectional dimensions as the functional design variables in order to minimize the total volume of the arch under given external pressure and geometrical constraints. Suitable optimality conditions are derived using the Pontryagin maximum principle. The solution requires a multimodal, even quadrimodal, formulation of the optimization problem to be introduced. Some detailed numerical results are presented and advantages connected with the assumption of two independent design variables are discussed.

1. INTRODUCTORY REMARKS

The optimal design of arches under stability constraints was the subject of many papers. The book by Gajewski and Zyczkowski (1988) gives a broad review of such papers published before 1986. The major part of them dealt with the unimodal formulation of the optimization problem and with one design variable only.

The need for the bimodal formulation of the optimization problem for arches was pointed out (Blachut and Gajewski, 1981a). A plane arch with an inextensible axis was considered and only in-plane buckling was admitted. If out-of-plane loss of stability of such an arch is admitted then, even bimodal formulation becomes insufficient. This was shown for a circular funicular arch with a rectangular cross-section where uni-, bi- or trimodal formulation was required in order to solve the optimization problem (Bochenek and Gajewski, 1986). That paper considered only one design variable namely either one of rectangle dimensions or both dimensions but with their ratio fixed.

This paper undertakes a new and more complicated problem, i.e. the optimal design of an arch for which both depth and width of a rectangular cross-section are treated as two independent design functions. The optimization problem of this kind was, for the first time, investigated for a compressed column with a rectangular cross-section which can buckle in two planes (Bochenek, 1987).

The arch is optimized against plane and spatial buckling and its axis is assumed to be inextensible. The influence of extensibility on the optimal design of a plane arch against in-plane buckling was previously considered (Blachut and Gajewski, 1981b). Variations were not taken into account whereas this problem for arches as multimodal was dealt with by Olhoff and Plaut (1983).

2. MATHEMATICAL DESCRIPTION

An elastic, plane and circular arch loaded by uniformly distributed radial pressure, shown in Fig. 1, is considered. The arch is assumed to be thin, slightly curved and its axis is treated as inextensible. Hence a momentless prebuckling state occurs (only the axial force $N_{z0} = -pR$ differs from zero) and the buckling state can be described by two sets of six first-order differential equations, one for in-plane and the other for out-of-plane loss of stability. These equations, the same as in Bochenek and Gajewski (1986), are re-written here in a more convenient matrix form

$$\frac{d}{ds} y_i^{(l)} = D_{ij}^{(l)} y_j^{(l)}$$

$$\frac{d}{ds} \bar{y}_i^{(m)} = \bar{D}_{ij}^{(m)} \bar{y}_j^{(m)}, \quad i,j = 1,2,\ldots,6; l = 1,2; m = 3,4 \tag{1}$$

where

$$\mathbb{Y}^{(l)} = \begin{bmatrix} v_l \\ M_{xl} \\ \gamma_l \\ M_{zl} \\ \alpha_l \\ K_{yl} \end{bmatrix}, \quad \mathbb{D}^{(l)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\varepsilon & -\varepsilon^2 p_l & 1 \\ 0 & 0 & 0 & \dfrac{\varphi_1\varphi_2}{C} & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\varphi_1}{B_x} & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbb{Y}}^{(m)} = \begin{bmatrix} w_m \\ M_{ym} \\ u_m \\ K_{xm} \\ \beta_m \\ N_{zm} \end{bmatrix}, \quad \bar{\mathbb{D}}^{(m)} = \begin{bmatrix} 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\varepsilon^2 p_m & 0 \\ -\varepsilon & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon \\ 0 & \dfrac{1}{B_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & 0 & 0 \end{bmatrix}. \tag{2}$$

All variables, i.e. components of displacement (u, v, w), angles of rotation (α, β, γ), increments of internal forces namely bending moments (M_x, M_y), twisting moment (M_z),

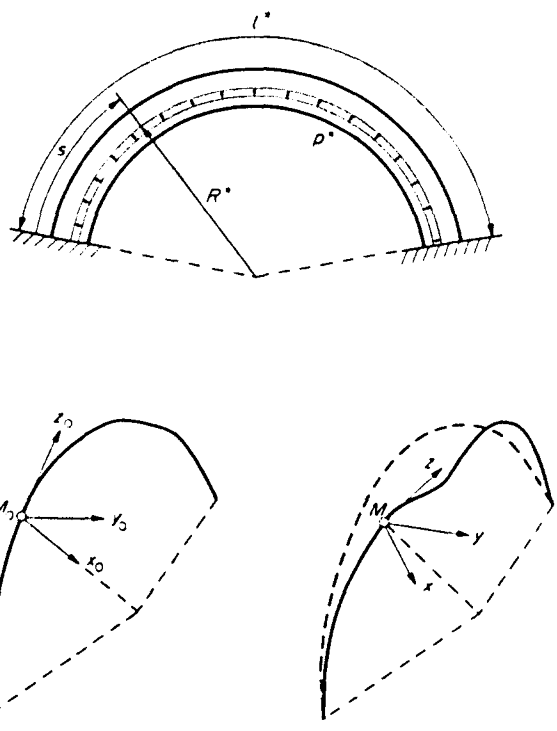

Fig. 1. A plane arch loaded by a constant external pressure.

shear forces (K_x, K_y), and axial force (N_z) are dimensionless and defined in relation to the coordinate system x_0, y_0, z_0—normal, binormal and tangent to the arch axis before buckling. Dimensionless B_x and B_y are the flexural rigidities, C the torsional rigidity, and $p_{l(m)}$ the external pressure. The definitions of these variables and additional parameters ε, φ_1, φ_2 are as follows:

$$u_m = \frac{u_m^*}{l^*},\quad v_l = \frac{v_l^*}{l^*},\quad w_m = \frac{w_m^*}{l^*},\quad M_{xl} = \frac{M_{xl}^* l^*}{B_{y0}^*},$$

$$M_{ym} = \frac{M_{ym}^* l^*}{B_{y0}^*},\quad M_{zl} = \frac{M_{zl}^* l^*}{B_{y0}^*},\quad K_{xm} = \frac{K_{xm}^* l^{*2}}{B_{y0}^*},$$

$$K_{yl} = \frac{K_{yl}^* l^{*2}}{B_{y0}^*},\quad N_{zm} = \frac{N_{zm}^* l^{*2}}{B_{y0}^*},\quad B_x = \frac{B_x^*}{B_{x0}^*},\quad B_y = \frac{B_y^*}{B_{y0}^*},$$

$$C = \frac{C^*}{C_0^*},\quad p_{l(m)} = \frac{p_{l(m)}^* R^{*3}}{B_{y0}^*},$$

$$\varepsilon = \frac{l^*}{R^*},\quad \varphi_1 = \frac{B_{y0}^*}{B_{x0}^*},\quad \varphi_2 = \frac{B_{y0}^*}{C_0^*},\quad l = 1, 2; m = 3, 4. \tag{3}$$

Dimensional quantities are marked with asterisks. B_{x0}^*, B_{y0}^*, C_0^* are certain constants to be defined later, l^* is the length of the arch, and R^* the radius of curvature of the undeformed axis. Symmetric and antisymmetric forms of in-plane and out-of-plane loss of stability connected with the lowest relevant critical loads $p_{l(m)}$ are considered. Indices l and m distinguish the following forms of buckling: $l = 1$ for symmetric out-of-plane, $l = 2$ for antisymmetric out-of-plane, $m = 3$ for symmetric in-plane, $m = 4$ for antisymmetric in-plane.

As regards the load behaviour in the course of buckling the considerations are confined to the case of fixed in space load direction. Moreover, arch ends are clamped so the boundary conditions for the state equations, eqns (1), are as follows:

$$\begin{aligned}
&\alpha_1(0) = \gamma_1(0) = v_1(0) = \alpha_1(\tfrac{1}{2}) = M_{z1}(\tfrac{1}{2}) = K_{y1}(\tfrac{1}{2}) = 0\\
&\alpha_2(0) = \gamma_2(0) = v_2(0) = \gamma_2(\tfrac{1}{2}) = v_2(\tfrac{1}{2}) = M_{x2}(\tfrac{1}{2}) = 0\\
&u_3(0) = w_3(0) = \beta_3(0) = \beta_3(\tfrac{1}{2}) = K_{x3}(\tfrac{1}{2}) = w_3(\tfrac{1}{2}) = 0\\
&u_4(0) = w_4(0) = \beta_4(0) = M_{y4}(\tfrac{1}{2}) = u_4(\tfrac{1}{2}) = N_{z4}(\tfrac{1}{2}) = 0.
\end{aligned} \tag{4}$$

Boundary conditions that distinguish symmetric and antisymmetric buckling modes are set up for $s = \frac{1}{2}$ due to the symmetry of the structure in the prebuckling state. Boundary conditions for $s = 0$ are given in the form common for both symmetric and antisymmetric modes. s denotes the independent variable measured along the arch axis.

The cross-section is assumed to be a rectangle and the dimensionless width b and depth h defined as

$$b = \frac{b^*}{\sqrt{A_0^*}},\quad h = \frac{h^*}{\sqrt{A_0^*}} \tag{5}$$

are treated as two independent functional design variables. The cross-sectional area A_0^* is chosen to satisfy

$$V_{\min}^* = A_0^* l^* \tag{6}$$

or

$$\int_0^1 \bar{b}\bar{h}\, \mathrm{d}s = 1 \tag{7}$$

where $V^*_{\min}$ is the minimal volume of the arch and $\bar{b}$ and $\bar{h}$ are the optimal control functions. Flexural rigidities, torsional rigidity and constants B^*_{x0}, B^*_{y0}, C^*_0 may be now presented in the form

$$B_x = hb^3, \quad B_y = bh^3, \quad C = hb^3\left[\frac{1}{3} - \frac{64}{\pi^5}\frac{b}{h}\,\mathrm{th}\left(\frac{\pi}{2}\frac{h}{b}\right)\right]$$

$$B^*_{x0} = \tfrac{1}{12}EA^{*2}_0, \quad B^*_{y0} = \tfrac{1}{12}EA^{*2}_0, \quad C^*_0 = GA^{*2}_0. \tag{8}$$

C has the same approximate form as given previously (Bochenek and Gajewski, 1986), and E and G are Young's and Kirchhoff's modulus, respectively.

3. THE OPTIMIZATION PROBLEM

The problem of optimal design is to determine two design functions $\bar{b}(s)$, $\bar{h}(s)$ that satisfy the state equations, eqns (1), with boundary conditions (4), normalization condition (7) and minimize the total volume of the arch under a given external load

$$\begin{aligned} V^* &\to V^*_{\min} \\ p^* &= \text{const.} \end{aligned} \tag{9}$$

In addition, geometrical constraints are imposed on both design functions

$$\begin{aligned} b_1 &\leqslant \bar{b}(s) \leqslant b_2 \\ h_1 &\leqslant \bar{h}(s) \leqslant h_2. \end{aligned} \tag{10}$$

With a view to obtain the necessary optimality condition the Pontryagin maximum principle is used. Introducing a new variable y_0 so as to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}y_0 = bh \tag{11}$$

with conditions $y_0(0) = 0$, $y_0(1) = 1$, adjoint state vectors

$$\begin{aligned} \Psi^{(l)} &= (\psi^{(l)}_v, \psi^{(l)}_{M_x}, \psi^{(l)}_{\gamma}, \psi^{(l)}_{M_z}, \psi^{(l)}_{\alpha}, \psi^{(l)}_{K_y}, \psi_0)^{\mathrm{T}} \\ \Psi^{(m)} &= (\bar{\psi}^{(m)}_w, \bar{\psi}^{(m)}_{M_y}, \bar{\psi}^{(m)}_u, \bar{\psi}^{(m)}_{K_x}, \bar{\psi}^{(m)}_{\beta}, \bar{\psi}^{(m)}_{N_z}, \psi_0)^{\mathrm{T}} \end{aligned} \tag{12}$$

and assuming, in general, four simultaneous modes of buckling corresponding to two lowest critical loads both for out-of-plane and in-plane loss of stability (quadrimodal formulation), the Hamiltonian may be written in the form

$$H = \psi_0 bh + \sum_{l=1}^{2} \psi^{(l)}_i D^{(l)}_{ij} y^{(l)}_j + \sum_{m=3}^{4} \bar{\psi}^{(m)}_i \bar{D}^{(m)}_{ij} \bar{y}^{(m)}_j. \tag{13}$$

It can be proved that the problem under consideration is self-adjoint. Hence eqn (13) takes the form

$$H = \psi_0 bh + \sum_{l=1}^{2} k_l\left[\varphi_1 \frac{M_{xl}^2}{B_x} + \varphi_1\varphi_2 \frac{M_{zl}^2}{C}\right] + \sum_{m=3}^{4} k_m \frac{M_{ym}^2}{B_y} + \cdots \tag{14}$$

In eqn (14) terms that are independent of b or h are omitted. k_l and k_m are nonnegative constants to be determined. If some of the k_l or k_m vanish a uni-, bi- or trimodal solution is obtained.

For two independent design variables b and h the necessary optimality condition takes the form of the following set of two equations:

$$\frac{\partial H}{\partial b} = 0, \quad \frac{\partial H}{\partial h} = 0. \tag{15}$$

The two transcendental algebraic equations obtained directly from eqns (14) and (15) can be replaced, after some algebra, by

$$\eta^2 = \frac{k_3 M_{y3}^2 + k_4 M_{y4}^2}{k_1 M_{x1}^2 + k_2 M_{x2}^2 + \varphi_2(k_1 M_{z1}^2 + k_2 M_{z2}^2)\,\dfrac{f - \eta \dfrac{df}{d\eta}}{f^2}}$$

$$b^6 = \frac{1}{\psi_0 \eta^4}\left\{k_3 M_{y3}^2 + k_4 M_{y4}^2 + \eta^2\left[3(k_1 M_{x1}^2 + k_2 M_{x2}^2) + \varphi_2(k_1 M_{z1}^2 + k_2 M_{z2}^2)\,\frac{3f - \eta \dfrac{df}{d\eta}}{f^2}\right]\right\} \tag{16}$$

where

$$\eta = \frac{h}{b}, \quad f(\eta) = \frac{1}{3} - \frac{64}{\pi^5}\frac{1}{\eta}\,\mathrm{th}\left(\frac{\pi\eta}{2}\right). \tag{17}$$

Only the first equation is transcendental with respect to η whereas the second is linear with respect to b^6.

4. NUMERICAL EXAMPLES

With the intent to solve the previously stated problem the iterative method proposed by Grinev and Filippov (1974), later used by other authors (Blachut and Gajewski, 1981a, b;

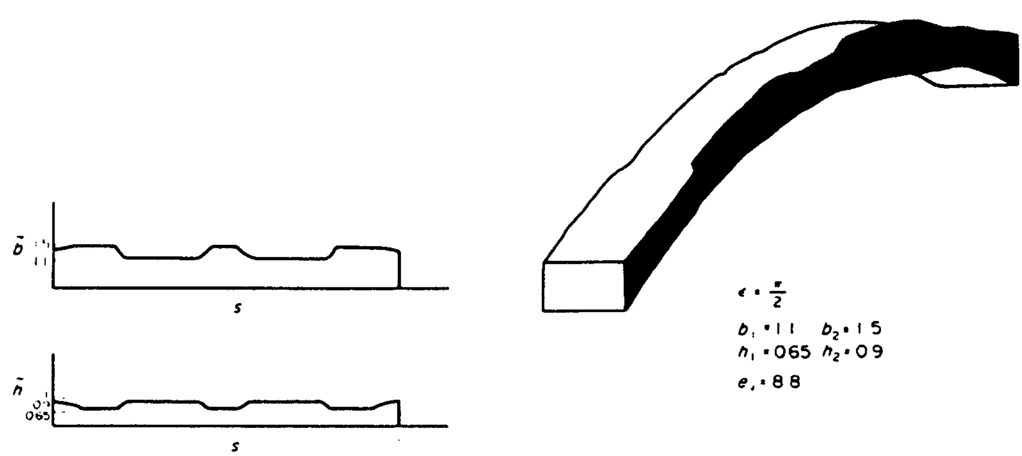

Fig. 2. The optimal arch for $\varepsilon = \pi/2$ (bimodal solution) and corresponding optimal functions $\bar{b}$, $\bar{h}$.

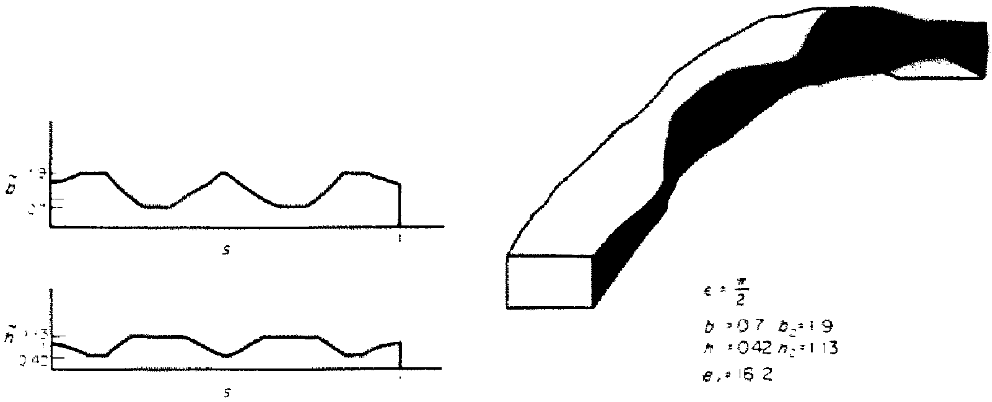

Fig. 3. The optimal arch for $\varepsilon = \pi/2$ (trimodal solution) and corresponding optimal functions $\tilde{b}$, $\tilde{h}$.

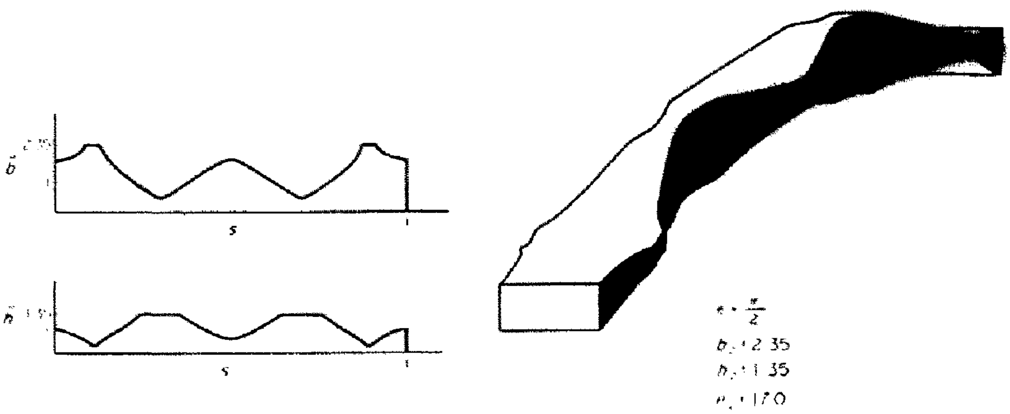

Fig. 4. The optimal arch for $\varepsilon = \pi/2$ (quadrimodal solution) and corresponding optimal functions $\tilde{b}$, $\tilde{h}$.

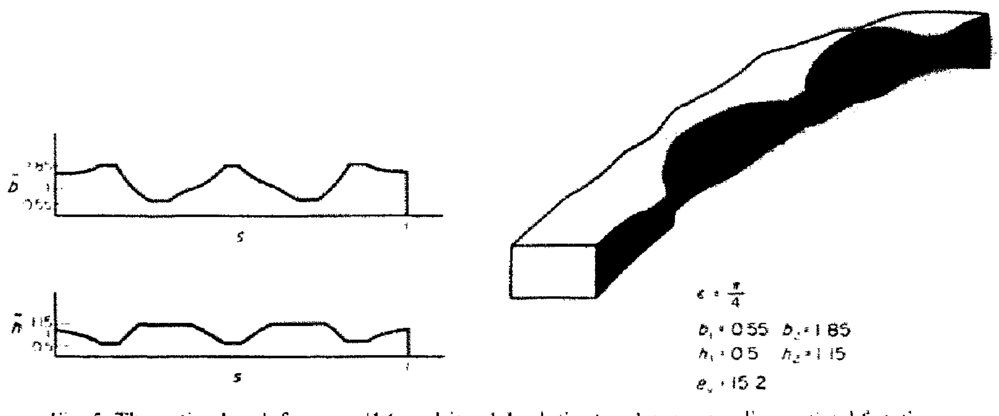

Fig. 5. The optimal arch for $\varepsilon = \pi/4$ (quadrimodal solution) and corresponding optimal functions $\tilde{b}$, $\tilde{h}$.

Bochenek and Gajewski, 1986) is applied. The method is treated as known and the details are not presented in this paper. Numerical integration of the state equations is performed step-by-step using the fourth-order Runge–Kutta method. Half the arch length is divided into one hundred parts.

Geometrical constraints b_1, b_2, h_1, h_2 are changed for given steepness ε of the arch. Optimization begins for the prismatic arch for which the lowest critical loads corresponding to symmetric out-of-plane and antisymmetric in-plane buckling have the same value—

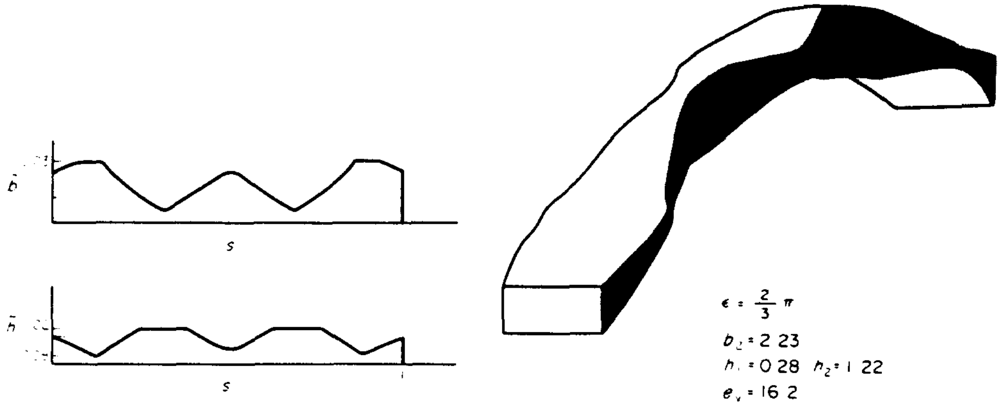

Fig. 6. The optimal arch for $\varepsilon = 2\pi/3$ (quadrimodal solution) and corresponding optimal functions $\bar{b}, \bar{h}$.

bimodal solution. For any set of constraints critical loads connected with the other buckling modes are calculated for the obtained optimal shape. If one of these loads has a lower value than the load for an optimal arch, the solution is no longer correct. The trimodal formulation which equalizes three critical loads has to be introduced. Analogically, in the case of an incorrect trimodal solution a quadrimodal formulation must be taken into consideration.

For $\varepsilon = \pi/2$ the starting prismatic arch has the rectangular cross-section with the ratio $b/h = 1.62$. The analysis of the results of a previous paper (Bochenek and Gajewski, 1986) leads to the conclusion that among all prismatic arches with rectangular cross-section loaded by the same buckling load the one with $b/h = 1.62$ has the lowest volume—the optimal bimodal prismatic arch. Furthermore, optimization with respect to only one design variable gives the volume reduction of about 10%.

For the purpose of comparison of those results with the ones obtained in the approach of this paper—for two independent design functions—detailed calculations for $\varepsilon = \pi/2$ were performed. The results are presented in Figs 2–4 where optimal functions $\bar{b}(s)$, $\bar{h}(s)$ and arch shape for certain geometrical constraints are shown. In addition, the values of the volume reduction coefficient c_v which is defined as

$$c_v = \frac{V^*_{\text{prism}} - V^*_{\text{min}}}{V^*_{\text{prism}}} 100\% \tag{18}$$

are given. It turns out that the volume reduction is greater than in the case of one design variable for the same starting prismatic arch. On the other hand optimal shapes are more complicated as to the mass distribution.

It is worth underlining that in order to pay attention on qualitative features of gained effects the $\bar{b}(s)$ and $\bar{h}(s)$ diagrams are presented with two different scales for abscissa and ordinate axes. Transverse dimensions are multiplied by 10, hence, to show real proportions between arch length and cross-sectional dimensions one has to divide them by 10. The arch is thin and the mass distribution changes slightly.

Figures 5 and 6 show optimal arches for steepness parameter $\varepsilon = \pi/4$ and $2\pi/3$. Presentation is confined to quadrimodal solutions only.

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